Spanning structures in graphs and hypergraphs

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Overview

In this talk we are interested in

- Graphs: undirected, simple graphs.
- k-uniform hypergraph $H = (V, E)$: $E \subseteq \binom{V}{k}$.
- Random graphs $\mathbb{G}(n, p)$: every pair of vertices is chosen independently with probability p.

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Spanning subgraphs in dense graphs

- (Dirac '52) every graph G of n vertices with $n \geq 3$ and $\delta(G) \geq n/2$ has a Hamiltonian cycle, that is, a cycle that passes through every vertex exactly once.

- (Hajnal–Szemerédi '70) for $t \geq 3$, every graph G of n vertices with $n \in t\mathbb{N}$ and $\delta(G) \geq (1 - 1/t)n$ has a K_t-factor. (Corradi–Hajnal for $t = 3$)

- (Alon–Yuster, '96, Komlós–Sárközy–Szemerédi '01) given a graph F of t vertices, every graph G of n vertices with $n \in t\mathbb{N}$ large and $\delta(G) \geq (1 - 1/\chi(F))n + C(F)$ has an F-factor.

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- (Komlós, '00, Shokoufandeh–Zhao '03) given a graph F of t vertices, every graph G of n vertices with n large and $\delta(G) \geq (1 - 1/\chi_{cf}(F))n$ has an F-tiling covering all but at most C vertices.

Theorem (Kühn–Osthus, '09)

For all F, the relevant parameter for F-factor is either $\chi_{cr}(F)$ or $\chi(F)$, and provide a dichotomy.

Theorem (H.–Treglown, '18+)

Given a graph G of n vertices and $\delta(G) \geq (1 - 1/\chi_{cf}(F) + o(1))n$, there is an algorithm that decides whether G has an F-factor in polynomial time.

Directed graphs, partite graphs, degree sequence...

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Randomly perturbed/augmented graphs

Randomly perturbed/augmented graphs: Add random edges to the deterministic (host) graph.

Theorem (Bohman–Frieze–Martin, '03)

Given any $\alpha > 0$, there exists $C = C(\alpha) > 0$ such that the following holds. Let G be a graph of n vertices with n large and $\delta(G) \geq \alpha n$. If we add Cn random edges to G, then the resulting graph whp. contains a Hamiltonian cycle.

- linearly many random edges are necessary
- If $\alpha = 0$, then we get the pure random model, where $Cn \log n$ random edges are needed.
- Compared with $\mathbb{G}(n, p)$, the 'saving' is a factor of $\log n$.

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F-factors in randomly perturbed graphs

- (Balogh–Treglown–Wagner, '18) If $\delta(G) \geq \alpha n$ and $p \geq Cn^{-1/d^*(F)}$, then $G \cup \mathbb{G}(n, p)$ contains an F-factor.

$$d^*(F) = \max\left\{\frac{e_H}{v_H - 1} : H \subseteq F, e_H > 0\right\}.$$

- (Böttcher–Montgomery–Parczyk–Person, '18+) bounded degree graphs F.

Question: [Treglown] For larger minimum degree, do we save more on the probability?
E.g., K_3-factors.

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Our result: K_r-factors

Theorem (H.–Morris–Treglown, 18++)

Let $2 \le k \le r-1$ be integers and $\gamma > 0$. Then there exists $C > 0$ such that if $\delta(G) \ge (1 - \frac{k}{r} + \gamma)n$ and $p \ge Cn^{-2/k}$, then $G \cup \mathbb{G}(n,p)$ has a K_r-factor with high probability.

- the condition on p is best possible up to the value of C
- the proof uses the absorbing method
  - uses the bipartite template initiated by Montgomery
  - for $k > r/2$, uses the lattice-based absorbing method

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Hamiltonicity in uniform hypergraphs

- (Krivelverich–Kwan–Sudakov, '16) perfect matchings and loose Hamiltonian cycles in k-uniform hypergraphs with minimum $(k-1)$-degree

Problem: [KKS] Extend this result to ℓ-cycles and minimum d-degree for all $1 \leq d, \ell \leq k-1$.

- (McDowell–Mycroft, '18+) ℓ-Hamiltonian cycles in k-uniform hypergraphs with minimum d-degree, $d \geq \max\{\ell, k-\ell\}$

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Powers of Hamiltonian cycles

Bounded degree spanning trees

Theorem (Krivelevich–Kwan–Sudakov, '17)

If $\delta(G) \geq \alpha n$, then $G \cup \mathbb{G}(n, C(\Delta)/n)$ whp. contains any spanning tree of maximum degree Δ.

Joos–Kim extended this result to trees with maximum degree $n/\log n$.

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If $\delta(G) \geq \alpha n$, then $G \cup \mathbb{G}(n, C(\Delta)/n)$ whp. contains all spanning trees of maximum degree Δ simutaneously.

In fact, we prove the result for a suitable expander graph with linearly many edges.

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Proof Sketch: Universality for bounded degree trees

- Decompose the tree: $T \supseteq T' \supseteq T_1$: T' has $(1-\epsilon)n$ vertices and T_1 has $\epsilon n \ll \alpha n$ vertices
- Pick $\epsilon' n$ disjoint random stars with Δ leaves, and finish the embedding of T_1 by the minimum degree condition
- Extend the embedding of T_1 to an embedding of T' by a result of Haxell
- Finish the embedding of T by the 'swapping' trick

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Thank you for your attention.